1 Algebra

1.1 Polynomials

- Binomial theorem
- Conjugate roots
- Symmetric polynomials (8, 32, 44, 45)
- Vieta's formulas (43, 44)
- Remainder theorem and factor theorem (51)
- Remainder problems (31)
- Rational root theorem
- Sum of coefficients (42)
- Dividing integer polynomials (33)
- Sum and difference of powers factorizations
- Simon's favorite factoring trick
- Polynomial long division (just write the coefficients)

1.2 Sequences, series, recurrences

- Arithmetic, geometric, arithmeto-geometric series
- Sum of squares, cubes
- Telescoping (46)
- Finite differences
- Solving linear recurrences (rare)
- Recurrences involving sums (rare)

1.3 Complex numbers

- Geometric multiplication and division
- DeMoivre's theorem
- Roots of unity (29, 30, 51)
- $|z|^2 = z\bar{z} = a^2 + b^2$

1.4 Other

- Maximum values with parabolas (47)
- $x \to ax$ scales your graph
- Functions with weird arguments (34)
- AM-GM inequality
- Logs: change of base, $\log_a b = 1/\log_b a$
- Nested radicals

2 Combinatorics

- Constructive counting (52, 56, 57)
- Block walking: two ways (11)
- The entries of Pascal's triangle are combinations (9)
- Rearrangements of MISSISSIPPI (10)
- Stars and bars (donuts and buckets) (11, 12, 54)
- "No two adjacent" problems (14, 13)
- Length of a sequence (make a correspondence) (19)
- Process trees (20)

- Symmetry factors (21)
- Case cutting (15, 21, 22, 23, 50)
- Compensating for overcounting (24, 49, 50)
- Principle of inclusion-exclusion (16, 53)
- Counting the zeros on the end of $n!$ (35)
- Geometric probability (28, 27, 58)
- Recasting the problem (17, 12, 48)
- Other (55)

3 Geometry

3.1 Triangles

- Special triangles
- $a/\sin A = b/\sin B = c/\sin C = 2R$
- $K = rs = \frac{1}{2}ab\sin\gamma = abc/4R = \sqrt{s(s-a)(s-b)(s-c)}$ (43)
- Medians cut each other in 2:1 ratios; triangles formed have 1/6th area
- Angle bisector theorem (41)
- Law of cosines and Stewart's theorem (41)
- Ceva's theorem
- Triangle areas via scaling (8)
- Mass points and area ratios
- Similar triangles generated by an altitude of a right triangle
- 13-14-15 triangle

3.2 Circles

- Tangent circles (internal or external): line connecting centers goes through tangency point
- Power of a point, including tangents (41)
- Inscribed angle theorem
- Angles inscribed in a semicircle are $90°$ (36)
- Angle-arc theorems
- Cyclic quadrilaterals: 2 important properties, plus $K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$
- Ptolemy's theorem

3.3 Strategies

- Always look for similar triangles.
- Look for congruent angles, especially from cyclic quadrilaterals and similar triangles.
- Questions asking for length ratios may require you to draw parallel lines.
- Trapezoid problems with two diagonals drawn usually involve similar triangles.
- What constraint haven't you used?
- Pythagorean theorem is everywhere. Extend lines and drop altitudes as needed.
- Under-constrained diagrams: there may be an inequality constraint. (26)
- Tangency problems: draw radii, use Pythagorean theorem. (37, 38, 39, 40)
- Look for isoceles triangles, due to perpendicular bisectors or equal angles.
- Isoceles triangles: drawing an altitude sometimes helps. (3)

3.4 Other

- Polygon areas by coordinates

- Pick's theorem: $K = I + B/2 - 1$ (rare)
- Golden pentagram (rare)
- Trigonometry: sum, difference, half-angle, double-angle formulas
- 3D Pythagorean theorem (25)

4 Number theory

- Divisibility tests for 3, 4, 8, 7, 9, 11
- Base conversions
- Number of divisors, sum of divisors
- Pythagorean triples
- Completing the square to solve Diophantine equations
- Playing with divisibility (4, 5)
- Chicken McNugget theorem ($ab - a - b$)
- Dividing congruences
- Linear diophantine equations (1, 6, 7)
- Diophantine equations involving rational expressions (2)
- Lattice points on the segment from $(0, 0)$ to (m, n)
- Chinese remainder theorem
- Fermat's little theorem (rare) (18)

5 Problems

1. Describe the integral solutions to the equation $317a + 241b = 9$.

2. For how many integral x is $(x + 49)/(x - 16)$ an integer?

3. Triangle ABC is isoceles with $AC = BC$ and $\angle ACB = 106°$. Point M is in the interior of the triangle so that $\angle MAC = 7°$ and $\angle MCA = 23°$. Find the number of degrees in $\angle CMB$. (2003 AIME #10)

4. Find the sum of all positive two-digit integers that are divisible by each of their digits. (2001 AIME I #1)

5. How many pairs of positive integers (a, b) are there such that $\gcd(a, b) = 1$ and
$$\frac{a}{b} + \frac{14b}{9a}$$
is an integer? (2007 AMC 12 B)

6. In a rectangular array of points, with 5 rows and N columns, the points are numbered consecutively from left to right beginning with the top row. Thus the top row is numbered 1 through N, the second row is numbered $N + 1$ through $2N$, and so forth. Five points, P_1, P_2, P_3, P_4, and P_5, are selected so that each P_i is in row i. Let x_i be the number associated with P_i. Now renumber the array consecutively from top to bottom, beginning with the first column. Let y_i be the number associated with P_i after the renumbering. It is found that $x_1 = y_2, x_2 = y_1, x_3 = y_4, x_4 = y_5$, and $x_5 = y_3$. Find the smallest possible value of N. (2001 AIME I #11)

7. \$100 is to be distributed among 100 men, women, and children so that each man receives \$5, each woman receives \$3, and each child receives 50 cents. How many men, women, and children are there? (There is more than one possibility.)

8. In triangle ABC, $AB = 13$, $BC = 15$ and $CA = 17$. Point D is on segment AB, E is on segment BC, and F is on segment CA. Let $AD = p \cdot AB$, $BE = q \cdot BC$, and $CF = r \cdot CA$, where p, q, and r are positive and satisfy $p + q + r = 2/3$ and $p^2 + q^2 + r^2 = 2/5$. Find the ratio of the area of triangle DEF to the area of triangle ABC. (2001 AIME I #9)

9. Simplify:

$$\binom{n}{k} - \binom{n-1}{k-1} - \binom{n-2}{k-1}$$

10. A spider must get dressed by putting a sock and a shoe on each of its 8 legs. On a given leg the sock must come before the shoe. All socks are identical, and all shoes are identical. In how many different orders could the spider put on its footwear? Express your answer as $p!/q^r$. (AMC)

11. A fair die is rolled four times. What is the probability that each of the final three rolls is at least as large as the roll preceding it? (2001 AIME I #6)

12. You have 100 (identical) coins which are each worth a billion dollars. Ten different failed companies would like a bailout, and you can give each of them between 0 and 100 coins (inclusive), as long as you don't go over 100 total. How many ways are there to distribute the coins?

13. A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers choose their spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park? (2008 AMC 12 B)

14. 10 children must be seated in a row of 30 chairs. The catch is that no two can sit next to each other, or they will chatter. How many possible seatings are there?

15. How many ways are there to color the faces of a regular tetrahedron using red, green, and blue? Each color may be used 0-4 times. Two colorings are considered to be the same if they are related by a rotation of the tetrahedron. What if there are four colors available?

16. How many numbers from 1 to 16^6 inclusive are either perfect squares or perfect cubes (or both)?

17. How many ways are there to write a sum of positive integers that equals 100? (For comparison, there are 4 sums that add to 3: $1+1+1$, $1+2$, $2+1$, and 3.)

18. What is the remainder when $98^{70} + 101^{52}$ is divided by 55?

19. How many distinct four-digit numbers are divisible by 3 and have 23 as their last two digits? (2003 10B #25)

20. A bag contains two red beads and two green beads. You reach into the bag and pull out a bead, replacing it with a red bead regardless of the color you pulled out. What is the probability that all beads in the bag are red after three such replacements? (2003 AMC 10 B #21)

21. The numbers 1 through 8 are randomly written on the faces of a regular octahedron so that each face contains a different number. What is the probability that no two consecutive numbers, where 8 and 1 are considered to be consecutive, are written on faces that share an edge? (2001 AIME #15)

22. A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible? (2001 AIME #14)

23. How many distinguishable ways are there to color the faces of a cube using 2 colors?

24. How many distinguishable ways can the faces of a regular octahedron be painted with 8 distinct colors? How many distinguishable ways can the faces of a regular dodecahedron be painted with 12 distinct colors?

25. See 2008 AIME I #15.

26. Let $ABCD$ be an isoceles trapezoid with AD parallel to BC whose angle at the longer base AD is $\pi/3$. The diagonals have length $10\sqrt{21}$, and point E is at distances $10\sqrt{7}$ and $30\sqrt{7}$ from vertices A and D, respectively. Let F be the foot of the altitude from C to AD. What is the distance EF? (2008 AIME I #10)

27. A circle of radius 1 is randomly placed in a 15-by-36 rectangle ABCD so that the circle lies completely within the rectangle. What is the probability that the circle will not touch diagonal AC? (AIME 2004 I #10)

28. A 1-inch toothpick is dropped onto a plane at a random position and orientation. An infinite number of parallel lines are drawn on the plane, each separated by 1 inch. What is the probability that the toothpick intersects a line?

29. Determine the roots of the polynomial
$$z^7 + z^5 + z^4 + z^3 + z^2 + 1 = 0.$$

30. Find the roots of the polynomial $P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$. (2004 AIME I #13)

31. Let $P(x)$ be a polynomial. When divided by $(x - 4)$, it has remainder 5. When divided by $(x - 5)$, it has remainder 6. What is its remainder when divided by $(x - 4)(x - 5)$?

32. What is the sum of the cubes of the roots of $2x^2 + 3x - 6$?

33. (2001 AMC 12) A polynomial of degree four with leading coefficient 1 and integer coefficients has two real zeros, both of which are integers. Which of the following can also be a zero of the polynomial?
$$\text{(A) } \frac{1+i\sqrt{11}}{2} \qquad \text{(B) } \frac{1+i}{2} \qquad \text{(C) } \frac{1}{2}+i \qquad \text{(D) } 1+\frac{i}{2} \qquad \text{(E) } \frac{1+i\sqrt{13}}{2}$$

34. Suppose that $P(x/3) = x^2 + x + 1$. What is the sum of all values of x for which $P(3x) = 7$? (2000 AMC10 #24)

35. 1000! ends in how many zeroes?

36. Semicircles are drawn outside sides AB and AD of square $ABCD$. Point E is the center of the square, and P and Q are points on the two semicircles (one on each). If P, A, and Q are collinear, $QA = 7$, and $AP = 23$, what is AE? (1994 ARML)

37. Two circles have radii 1 and 7, and one of their common internal tangents is perpendicular to a common external tangent. Compute the distance between their centers. (NYSML)

38. Segment AB is the diameter of a semicircle; O is the midpoint of AB. Circle P is tangent to AB at O and to the semicircle. Circle Q is tangent to AB, to the semicircle, and to circle P. If $OB = 1$, what is the radius of circle Q? (ARML)

39. Two congruent circles are inscribed in a 3-4-5 triangle so that each touches the hypotenuse, a leg, and the other circle. What is their radius?

40. Two circles are externally tangent at P. One of their other common tangents touches the circles at A and B. A line is drawn through P parallel to AB, meeting the circles again at C and D. If the radii of the circles are 2 and 18, find the distance between the midpoints of AB and CD. (ARML)

41. Triangle ABC is inscribed in a circle and the bisector of angle ABC meets the circle at P. If $AB = 6$, $BC = 8$, and $AC = 7$, compute BP. (ARML)

42. If $P(x)$ is a polynomial in x, and $x^{23} + 23x^{17} - 18x^{16} - 24x^{15} + 108x^{14} = (x^4 - 3x^2 - 2x + 9)P(x)$, compute the sum of the coefficients of $P(x)$. (1989 ARML)

43. The equation $x^3 - 4x^2 + 5x - 1.9$ has real roots r,s, and t. Find the area of the triangle with sides r, s, and t. (Mandelbrot, Jan. 1997)

44. Determine $(r+s)(s+t)(t+r)$ if r, s, and t are the three roots of the polynomial $x^3 + 9x^2 - 9x - 8$. (Mandelbrot, Oct. 1998)

45. If r and s are the roots of $x^2+x+7=0$, compute the numerical value of $2r^2+rs+s^2+r+7$. (NYSML 1991)

46. Find the following sums:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \qquad \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \qquad \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$$

47. Vanna White is selling vowels. If she charges \$10 per vowel, 60 people will buy one. For every additional \$1 she charges, 3 fewer people will buy one. How much should Vanna charge to maximize her profit?

48. Suppose you have two distinguishable flagpoles and 19 flags, of which 10 are identical blue flags, and 9 are identical green flags. In how many distinguishable ways can all the flags be arranged on the flagpoles so that each flagpole has at least one flag and no two green flags on either pole are adjacent? (2008 AIME II #12)

49. In how many distinguishable ways can 5 red unit cubes and 5 blue unit cubes be glued together to form a $10 \times 1 \times 1$ solid? How about 4 red and 4 blue?

50. In how many distinguishable ways can 5 identical red beads and 5 identical blue beads be arranged on a circular ring?

51. Let
$$f(x) = x^4 + x^3 + x^2 + x + 1.$$
Find the remainder when $f(x^5)$ is divided by $f(x)$.

52. How many ways are there to select 5 cards from a standard 52-card deck so that
    (a) all cards have the same suit
    (b) three cards share a rank, and the other two share a rank
    (c) four cards share a rank
    (d) no two cards have the same rank

    (Rank means one of the following: ace, 2, 3, 4, ..., 9, 10, jack, queen, or king.)

53. How many positive integers less than 1000 are relatively prime to 360?

54. Ten dice are rolled. What is the probability that the sum of the rolls is 20?

55. The faces of a cube are randomly colored using red and blue. What is the probability that the cube can be placed on a table so that the top and bottom faces are the same color?

56. Two diagonals are selected at random from a regular n-sided polygon so that they do not share any endpoints. What is the probability that they intersect?

57. How many numbers between 1000 and 9999 have distinct digits $ABCD$ with $A < B, B > C, C < D$? (AIME)

58. A meter stick is cut in two random places, producing 3 segments. What is the probability that the longest segment is more than 50 centimeters long?